

# Spectral Equivalence of Bosons and Fermions in One-Dimensional Harmonic Potentials

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*February, 2000*

**ABSTRACT:** Recently, Schmidt and Schnack (Physica A260 479 (1998)), following earlier references, reiterate that the specific heat of  $N$  non-interacting bosons in a one-dimensional harmonic well equals that of  $N$  non-interacting fermions in the same potential. We show that this peculiar relationship between heat capacities results from a more dramatic equivalence between bose and fermi systems. Namely, we prove that the excitation spectrums of such bose and fermi systems are *spectrally equivalent*. Two complementary proofs of this equivalence are provided, one based on an analysis of the dynamical symmetry group of the  $N$ -body system, the other using a combinatoric analysis.

**I. Introduction:** With the advent of dilute atomic BEC<sup>[1,2,3]</sup> and, recently, nearly degenerate dilute atomic fermi gas<sup>[4,5]</sup>, there is renewed interest in understanding aspects of quantum many-body theory in inhomogeneous (in particular, harmonically trapped) systems. Since trapped, cooled atoms have properties that are, in principle, controllable to a degree unavailable in other systems (e.g., clusters and nuclei), they present new opportunities to study quantum mechanics and many-body theory.

Although the ultra-cold dilute atomic gas systems are large compared to the coherence lengths, they are not homogeneous, due to the fact they are generally trapped in a (nearly) harmonic potential. In many of these systems the interparticle forces are significant. In this note, however, we ignore the interactions between atoms, with the motivation being to understand better the thermodynamic properties of  $N$  trapped *non-interacting* bosons and fermions. Recent work<sup>[6,7]</sup> describes strange relations between the equilibrium thermodynamics of these two systems. It was shown, for example, that the heat capacity (as a function of temperature) of  $N$  noninteracting bosons in a one-dimensional harmonic potential is the same as that of  $N$  noninteracting fermions in an identical potential. The respective partition functions for these systems are likewise closely related (see Ref. [7]).

These "coincidences" provide hints that a deeper underlying connection exists between bose and fermi gases in a harmonic well. In particular, the heat capacity and partition functions as functions of the inverse temperature  $\beta$  can be thought of as an "imaginary time" continuation of a fourier transform of the spectrum. The fact that the heat capacities are the same for all temperatures suggests that there should be a state-for-state, level-for-level correspondence between these non-interacting many-body bosonic and fermionic systems. We show that this is indeed the case, and below describe two independent proofs of the spectral equivalence of excitations in these systems. The first of these arguments relies on the dynamical symmetry group properties of these systems, while the other is based on combinatoric methods.

**II. Dynamical symmetry group approach:** Consider  $N$  non-interacting bosons or fermions in a harmonic well. In either case, the energy differences between energy levels is an integer multiple of  $\hbar\omega$ . Consequently, to analyze the spectral properties of each system it suffices to study the multiplicity per (total energy) level. Below we will show that the entire spectrum of both the bosonic and fermionic systems are isomorphic up to an overall energy shift.

Classically, a system of  $N$  non-interacting particles in a 1-d harmonic potential is identical to that of a single particle in an  $N$ -dimensional isotropic harmonic potential. The system thus has an obvious spatial  $O(N)$  symmetry we call "angular momentum". However, it is apparent with more introspection that the system possesses a much larger dynamical symmetry group. Orbits in the  $N$ -dimensional isotropic harmonic potential do not precess. In analogy with the Kepler problem, we say that there is a conserved Runge-Lenz vector (which may be thought of as the axis of the orbit in configuration space), and we thus expect the symmetry group to be enlarged.

Since we will be interested in the quantization of the system, we describe the dynamical symmetry enlargement through the operators of the associated quantum theory. To simplify notation, take  $\hbar\omega = 1$  throughout. Label the raising and lowering operators for the bosonic theory  $a_i^\dagger, a_i$  with  $i = 1, \dots, N$ . The canonical commutation relations (for the bosonic case) are  $[a_i, a_j^\dagger] = \delta_{ij}$ . The many-body hamiltonian operator of this noninteracting system is  $H = \sum_i a_i^\dagger a_i + \epsilon$ , where  $\epsilon$  is an overall constant.

We call the space of the eigenvalues of the  $a_i^\dagger a_i$  the state space. Equivalently, the state space is the integer lattice in the  $(+, \dots, +)$  quadrant of  $N$ -dimensional Euclidean space. State space is

not the Fock space, but is a useful auxiliary space from which we will construct the Fock space, and so we discuss its properties. Let  $e_i$  be orthonormal unit basis vectors in this Euclidean space associated with the eigenvalues of  $a_i^\dagger a_i$ . We name several distinguished vectors in this space, namely the *level* vector  $k = \sum_i e_i$  and the *root* vectors  $l_i = e_i - e_{i+1}$  for  $i = 1, \dots, N-1$ . We also define a spanning set of *weight* vectors  $r_i$  via  $(r_i, l_j) = \delta_{i,j}$  together with  $(r_i, k) = 0$ .

Note that the operators associated with  $l_i$ , namely  $a_i^\dagger a_i - a_{i+1}^\dagger a_{i+1}$ , are independent, and commute with each other (being all diagonal) and with the hamiltonian. The hamiltonian corresponds to the level vector. Furthermore, to each pair of particles  $l \neq j$ , there is an associated  $su(2)$  subalgebra generated by  $\{a_l^\dagger a_l - a_j^\dagger a_j, a_l^\dagger a_j + a_j^\dagger a_l, i(a_l^\dagger a_j - a_j^\dagger a_l)\}$ . The application of the second or third operators in the above  $su(2)$  subalgebra “shift” the first operator’s eigenvalue by a combination of root vectors. Finally, note that the matrix of inner products  $M_{ij} = (l_i, l_j)$  of the root vectors is exactly the cartan matrix of  $su(N)$ . Thus, we have identified the dynamical symmetry group of this system generated by the (trace-free part of the) products of  $a_i^\dagger a_j$  to be  $su(N)$ .

We now construct the Fock space for both fermions and bosons from the state space by realizing the respective anti-symmetrizations and symmetrizations of the multi-particle Fock states as linear combinations of states in the state space that lie on the same Weyl group orbit. We make this correspondence precise with the following observations. Each state in the state space can be thought of as a particular product of single particle states, its coordinates (the components of the  $l_i$  are integers) are simply the harmonic oscillator level of each particle. Constructing the multi-particle state associated with that product of single particle states consists of combining all the states from single-particle label permutations. The permutation group  $S_N$  is generated by primitive transpositions  $(\dots, n_i, n_{i+1}, \dots) \rightarrow (\dots, n_{i+1}, n_i, \dots)$ . Each of these primitive transpositions acts as a Weyl reflection (acting on all the roots) about the hyperplane perpendicular to the root  $l_i$ .

Thus, the Weyl group,  $\mathcal{W}$  of the symmetry algebra  $su(N)$  is exactly the group of permutations of the single particle states that make up the many-body state. Each element of the Weyl group preserves the level  $k$ . We specify a many body state through an assignment of a highest weight vector  $r$  and level  $s$  (a natural number) for which  $r + \frac{sk}{N}$  is a vector in the  $(+, \dots, +)$  quadrant (boundaries included). Explicitly, in terms of the vectors in the state space, the bosonic many-body Fock space has the basis  $\Psi_{r,s}^{boson}$

$$\Psi_{r,s}^{boson} = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{W}} \left| \frac{s}{N} k + \sigma \cdot r \right\rangle$$

whereas the basis of the fermionic many-body Fock space is

$$\Psi_{r,s}^{fermion} = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{W}} (-1)^{sgn(\sigma)} \left| \frac{s}{N} k + \sigma \cdot r \right\rangle$$

where the  $sgn(\sigma)$  is 1 if  $\sigma$  is an even permutation and -1 if it is an odd permutation. Note that according to this definition, only  $r$  vectors from the interior of the Weyl chamber are associated with a fermionic many-body state.

Succinctly stated, the multi-particle permutation symmetry of quantum mechanics maps the single particle states of state space into the highest weight space of the symmetry algebra  $su(N)$ . For bosons, the map covers the entire Weyl chamber (including the lattice points in the bounding hyperplanes) at each level. For fermions, the map covers only the interior lattice points of the Weyl chamber. Additionally, due to the constraint that  $r + \frac{sk}{N}$  is in the  $(+, \dots, +)$  quadrant, at each level there are of course only a finite number of highest weight candidates.

The vector  $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$  (half the sum of positive roots) translates the vacuum of the bosonic Fock space to that of the fermionic Fock space at each level. Note also that  $\rho$  is thus orthogonal to the level vector  $k$ . It can be combined with the level vector to constitute a one-to-one map between the spectrum of the  $N$  boson and  $N$  fermion systems. Note that translation by the vector  $\Gamma = (0, 1, 2, \dots, N-1)$  is precisely that map, and that  $\Gamma = \rho + \frac{N-1}{2}k$ . Note further that  $\Gamma$  has level  $\Gamma \cdot k = N(N-1)/2$ , which is precisely the ground state energy shift between the bosonic and fermionic system. Geometrically,  $\Gamma$  is the smallest lattice vector that translates the lattice points in the bounding hyperplanes entirely into (the subset of) the interior of the Weyl chambers at each level.

**III Combinatoric approach:** The spectral equivalence of one-dimensional, noninteracting harmonically-trapped bosonic and fermionic gases can also be understood through a straightforward combinatoric argument.

In a system of  $N$  noninteracting particles (bosons or fermions) in a harmonic well let the energy level of the  $i$ th particle be specified by the integer  $e_i$ , with  $E = \sum_{i=1}^N e_i$  the total energy of the system. (Note: In writing the energy  $e_i$  as an integer, we are, as before, setting  $\hbar\omega = 1$ , and for notational convenience are ignoring the constant  $1/2$  associated with the single-particle ground state energy.) Clearly there are many different microconfigurations possessing the same total energy  $E$ ; we let  $G_N(E)$  denote the multiplicity of states with fixed energy  $E$ . We will show that the multiplicity functions for bosons and fermions are equivalent. More precisely, we show that  $G_N^{boson}(E) = G_N^{fermion}(E + N(N-1)/2)$ , indicating that the multiplicities for the bose and fermi cases are identical provided each is measured relative to its respective ground state energy (i.e., 0 for bosons and  $N(N-1)/2$  for fermions). This equivalence is sufficient to prove the spectral equivalence of the excitation spectrum.

We begin with the bose case. We imagine ordering the  $N$  particles from lowest energy to highest ( $e_1, e_2, \dots, e_N$ ). The energy of the lowest-energy particle ( $e_1$ ) can range from zero up to a maximum value of  $[E/N]$ , where the brackets  $[ ]$  denote the integer part of the expression enclosed. (It is readily seen that if the energy of the *lowest*-energy particle were to exceed this maximum value, then the sum of the energies of the  $N$  individual particles would exceed the total specified energy  $E$  of the system.)

For a fixed  $e_1$ , the remaining energy  $E - e_1$  must be divided up among  $N - 1$  particles. So the possible values of  $e_2$ , which represents the lowest energy among the remaining  $(N - 1)$  particles, can range from  $e_1$  to  $\left\lceil \frac{E - e_1}{N - 1} \right\rceil$ . (As before, it is clear that if  $e_2$  went outside this range, then the sum of the energies of the  $N - 1$  particles would exceed the prescribed value  $E - e_1$ .)

Proceeding in this fashion, we see that

$$G_N^{boson}(E) = \sum_{e_1=0}^{[E/N]} \sum_{e_2=e_1}^{\left\lceil \frac{E-e_1}{N-1} \right\rceil} \dots \sum_{e_{N-1}=e_{N-2}}^{\left\lceil \frac{E-e_1-e_2-\dots-e_{N-2}}{2} \right\rceil} 1.$$

A similar argument is used to construct the multiplicity function for the fermionic case. The fundamental distinction stems from the additional constraint that two fermions cannot occupy the same energy orbital, which in turn modifies the lower and upper bounds in the above summations, as we now describe. Consider first the lower bounds. From the exclusion principle, it immediately follows that the lower (fermionic) bounds must take the form  $e_i = e_{i-1} + 1$ . The upper limits are

found by noting that for a system of  $N$  fermions with total energy  $E$ , the energy of the lowest-energy fermion cannot exceed  $\left\lceil \frac{E - \frac{N(N-1)}{2}}{N} \right\rceil$ , as a straightforward calculation reveals. Consequently, we find

$$G_N^{fermion}(E) = \sum_{e_1=0}^{\left\lceil \frac{E - \frac{N(N-1)}{2}}{N} \right\rceil} \sum_{e_2=e_1+1}^{\left\lceil \frac{E - e_1 - \frac{(N-1)(N-2)}{2}}{N-1} \right\rceil} \dots \sum_{e_{N-1}=e_{N-2}+1}^{\left\lceil \frac{E - e_1 - e_2 - \dots - e_{N-2} - \frac{(2)(1)}{2}}{2} \right\rceil} 1.$$

Expressed in this manner, the equivalence of  $G_N^{boson}(E)$  and  $G_N^{fermion}(E + N(N-1)/2)$  is now revealed through the following key coordinate transformation: In the fermionic summations above, introduce new coordinates  $\hat{e}_i = e_i - i + 1$ . We claim that this will transform the fermionic sum into the corresponding bose sum. (Note: in the context of the preceding analysis, this coordinate change serves to relate the interior lattice points of the Weyl chamber (fermionic case) to the entire Weyl chamber (bose case), that is, it is simply the translation by the vector  $\Gamma$ ) To see that this transformation achieves the desired result, first observe that under this transformation, the lower bounds in the fermionic summations ( $e_{i+1} = e_i + 1$ ) become ( $\hat{e}_{i+1} = \hat{e}_i$ ), just as in the bose case. Meanwhile, it is not difficult to verify that the upper limits in the fermionic summations

$$e_{i+1} = \left\lceil \frac{E - e_1 - e_2 - \dots - e_i - \frac{(N-i)(N-i-1)}{2}}{N-i} \right\rceil$$

now take the form

$$\hat{e}_{i+1} = \left\lceil \frac{E - \frac{N(N-1)}{2} - \hat{e}_1 - \hat{e}_2 - \dots - \hat{e}_i}{N-i} \right\rceil,$$

which, again is the same as for the bosonic case (once we shift by the fermionic ground state energy  $E \rightarrow E + N(N-1)/2$ ).

This equivalence between the bosonic and fermionic multiplicity functions proves that the excitation spectrum of one-dimensional harmonically trapped  $N$  non-interacting bosons is identical to that of  $N$  non-interacting fermions.

**IV. Remarks and Conclusion:** Although the excitation spectra of the fermi and bose systems are identical, these systems are not related by an obvious supersymmetry. There may, however, exist a connection associated with the fermionic representation of affine lie algebra characters as described in Refs.[9,10,11]. Lastly, we observe that the recent work of Schmidt and Schnack<sup>6,7</sup> indicates that the specific heats of similar bose and fermi systems in higher spatial dimensions (specifically, odd dimensions) might also be equivalent, just as for the one-dimensional case considered here. However, preliminary work suggests that spectral equivalence does not persist in higher (odd) dimensions.

**V: Acknowledgments** This research was supported in part by Research Corporation Cottrell Science Award #CC3943 and in part by the National Science Foundation under grants PHY 94-07194 and EPS-9874764.

## Bibliography

1. M. H. Anderson, J. R. Ensher, M. R. Mathews, C. E. Weiman and E. A. Cornell, *Science* **269**, 198 (1995).
2. K. B. Davis *et. al.*, *Phys. Rev. Lett.* **75**, 3969 (1995).
3. C. C. Bradley, C. A. Sackett and R. G. Hulet, (to be published).
4. B. DeMarco and D. S. Jin, *Phys. Rev. A* **58**, R4267 (1998).
5. B. DeMarco, Bohm, Burke, Holland, D. S. Jin, *Phys. Rev. Lett.*, **82**:(21) 4208, (1999), cond-mat/9812350.
6. H.-J. Schmidt and J. Schnack, “Thermodynamic fermion-boson symmetry in Harmonic Oscillator Potentials,” cond-mat/9810036
7. H.-J. Schmidt and J. Schnack, *Physica* **A260** 479, (1998) cond-mat/9803151
8. J. E. Humphreys, “Introduction to Lie Algebras and Representation Theory,” Springer-Verlag, New York, 1972
9. R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, *Phys. Lett.* **B307** (1993) 68-76, hep-th/9301046
10. E. Melzer, *Int. J. Mod. Phys.* **A9** (1994), 1115-1136, hep-th/9305114
11. E. Bauer and D. Gepner, *Phys. Lett.* **B372** (1996) 231-235, hep-th/9502118